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Double Superpotential and Generation of New Solutions of Stationary Axisymmetric Gravitational Fields

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Abstract In this paper COX's superpotential is extended to double complex form firstly, then how to get hyperbolic complex superpotential from the known ordinary complex superpotential is discussed, finally, new solutions of stationary axisymmetric gravitational fields are shown with a few specific cases.

Keywords Double complex number · Stationary axisymmetric gravitational field · Superpotential

1 Introduction

As one of Abel complex numbers, the hyperbolic complex function was barely applied directly to the field of physics though mathematicians have already done a lot of research on it in 19 century. It was not until 1983, when Kunstattar, Maffat and Malzan [1] proposed that the metric of spacetime manifold should be adopted as hyperbolic complex numbers in order to deal with the nonsymmetric gravitation theory that hyperbolic complex numbers were applied to physics for the first time. In recent years, hyperbolic complex numbers and hyperbolic complex group have their large applications in every field of physics. Cox [2] once gave a special method to solve Ernst equation by introducing superpotential so as to get a group of new solutions of stationary axisymmetric gravitational field. Nevertheless, Cox's results were certainly restricted as he only used the ordinary complex number. As a matter of fact, Zaizhe Zhong [3, 4] has found out a double complex function method and established double complex Ernst equation. Hence, the solutions of stationary axisymmetric gravitational fields are certainly to be obtained in couples at the same time. If so, a natural

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question comes out: Could the Cox superpotential method be used to solve the double complex Ernst equation? The above problem will be discussed in the present paper. We give double superpotential similarly and show double complex Ernst equation with Cox form correspondingly. With this equation, we generate hyperbolic complex superpotential from the known ordinary complex superpotential. It means that new axisymmetric solutions can be obtained from a few known solutions of stationary symmetric gravitational fields. Interestingly, we find out a very special situation, that is even the gravitational field of ordinary complex superpotential is nonphysical the dual hyperbolic complex Ernst potential maybe still correspond to physical field solution. Thus, clearly, it will be very difficult to get this kind of field solutions with other methods. In this paper, first we discuss double complex Ernst equation and the establishment of double complex superpotential, then we discuss the generation and calculation of dual superpotential with a few specific cases.

2 Double Complex Ernst Equation and Double Superpotential

According to the argument of Zaizhe Zhong [3], the vacuum Einstein's equation can be changed into the double Ernst equation

$$\mathrm{Re}(E(J))\nabla^2 E(J) = \nabla E(J)\cdot\nabla E(J), \tag{1}$$

where $E(J)$ is the double complex Ernst potential ($J = i$, $i^2 = -1$; $J = \epsilon$, $\epsilon^2 = 1$)

$$E(J) = F(J) + J\cdot\Omega(J), \tag{2}$$

$F(J) = F(\rho, z; J)$, $\Omega(\rho, z; J)$ are both double complex functions. If $E(J) = F(J) + J\cdot\Omega(J)$ is a double complex solution of (1), we can get the dual solutions of stationary axisymmetric vacuum as

$$\begin{aligned}(f,\omega) &= (F_C, V_{F_C}(\Omega_C)),\\ (\hat{f},\hat{\omega}) &= (T(F_H), \Omega_H).\end{aligned} \tag{3}$$

where transformations T and V are equivalent to the N-K substitution [5]

$$\begin{aligned}&T : f \to T(f) = \rho f^{-1},\\ &V : (f,\psi) \to V_f(\psi) = \omega,\\ &\partial_\rho\psi = \rho^{-1} f^2 \partial_z\omega, \qquad \partial_z\psi = -\rho^{-1} f^2\partial_\rho\omega.\end{aligned} \tag{4}$$

$$\begin{aligned}\gamma_\rho &= \{\rho F_C^{-2}(F_{C\rho}^2 - F_{Cz}^2) - \rho^{-1}F_C^2[(V_{F_C}(\Omega_C))_\rho^2 - (V_{F_C}(\Omega_C))_z^2]\}/4,\\ \gamma_z &= \{\rho F_C^{-2}F_{C\rho}F_{Cz} - \rho^{-1}F_C^2[V_{F_C}(\Omega_C)]_\rho[V_{F_C}(\Omega_C)]_z\}/2,\\ \hat{\gamma}_\rho &= [F_H^2(F_{H\rho}^2 - F_{Hz}^2)/\rho - F_H^2(\Omega_{H\rho}^2 - \Omega_{Hz}^2)/\rho]/4,\\ \hat{\gamma}_z &= [F_H^2 F_{H\rho}F_{Hz}/\rho - F_H^2\Omega_{H\rho}\Omega_{Hz}/\rho]/2.\end{aligned} \tag{5}$$

Noting that (1) actually is

$$\begin{aligned}&\nabla_{(\dot{C})}[F_C^{-2}\nabla_{(C)}(F_C^2 + \Omega_C^2)] = 0,\\ &\nabla_{(\dot{C})}[F_C^{-2}\nabla_{(C)}\Omega_C] = 0;\\ &\nabla_{(\dot{H})}[F_H^{-2}\nabla_{(H)}(F_H^2 - \Omega_H^2)] = 0,\\ &\nabla_{(\dot{H})}[F_H^{-2}\nabla_{(H)}\Omega_H] = 0.\end{aligned} \tag{6}$$

For double line element

$$d\sigma^2(J) = h(J)_{AB} dx^A dx^B \tag{7}$$

define the dot product of gradient operator $\nabla_{(J)}$ for any double complex number $G(J)$

$$\nabla_{(j)} \cdot G(J) = \frac{1}{\sqrt{h(J)}} \frac{\partial}{\partial x^B} [\sqrt{h(J)} h^{AB}(J) G(J)_{,A}], \tag{8}$$

where $h(c)_{AB} = h_{AB}$ is determined by (f, ω, γ); $h(H)_{AB} = \hat{h}_{AB}$ is determined by $(\hat{f}, \hat{\omega}, \hat{\gamma})$. Hence, (6) can be expressed as

$$\begin{aligned} &\nabla_{(j)}[F^{-2}(J)\nabla_{(j)}(F^2(J) - J^2 \cdot \Omega^2(J))] = 0, \\ &\nabla_{(j)}[F^{-2}(J)\nabla_{(j)}\Omega(J)] = 0. \end{aligned} \tag{9}$$

If the double solutions $(F(J), \Omega(J))$ of the above equation are known, $(f, \omega, h_{AB})$and $(\hat{f}, \hat{\omega}, \hat{h}_{AB})$ can be determined by (3) and (4). The characteristic of (9): first it is double, every solution will be a couple of dual solutions; secondly, it is covariant, that is its form does not change in other coordinate frames, which is of great importance as we use super-potential method to solve the Ernst equation.

Choosing the double intrinsic coordinate $(x^1(J), x^2(J)) = (F^2(J) - J^2\Omega^2(J), \Omega(J))$, (9) can be expressed as

$$\begin{aligned} &[\rho(J)F^{-2}(J)\bar{h}^{AB}(J)_{,B}]_{,A} = 0 \quad (A, B = 1, 2), \\ &\det \bar{h}^{AB}(J) = 1. \end{aligned} \tag{10}$$

where $\bar{h}^{AB}(J) = h^{1/2}(J)h^{AB}(J)$. The most common solution of (10) is

$$\begin{aligned} &\bar{h}^{AB}(J) = \rho^{-1}(J)F^2(J)\epsilon^{AC}\epsilon^{BD}K(J)_{,CD} = \rho^{-1}(J)F^2(J)K(J)^{(AB)}, \\ &\det(K(J)_{,AB}) = \rho^2(J)F^{-4}(J). \end{aligned} \tag{11}$$

$K(J)$ is the double super potential. Combining (9) we have

$$\begin{aligned} &F^2(J) = x_J^1 + J^2(x_J^2)^2, \qquad \bar{h}_{AB}(J) = K(J)_{,AB}/\Delta(J), \\ &\Delta(J) \equiv \det{}^{1/2}[K(J)_{,AB}], \qquad \rho(J) = \Delta(J)F^2(J), \\ &\bar{\nabla}^2_{(J)}\rho(J) = 0, \qquad \overline{\nabla^2} = \bar{h}^{AB}\bar{\nabla}_A\bar{\nabla}_B, \end{aligned} \tag{12}$$

where $\bar{\nabla}_{(J)}$ is the covariant derivative with respect to $\bar{h}^{AB}(J)$. If double super potential $K(J)$ has been got as

$$\begin{aligned} &\omega_{,A} = \rho F_C^{-2}\bar{h}^{AB}\epsilon^{B2} = K_{,1A}, \\ &\omega = K_{,1} + \text{const.} \end{aligned} \tag{13}$$

The constant term can be deleted: by the coordinate conversion $t \to t + \alpha\phi$, setting

$$\omega = K_{,1}, \tag{14}$$

the form of the metric is

$$ds^2 = F_C(dt + K_{,1}d\phi)^2 - F_C^{-1}(e^{2\gamma}\bar{h}_{AB}dx^A dx^B + \rho^2 d\phi^2). \tag{15}$$

γ is given by the following equation

$$\bar{\nabla}_A\bar{\nabla}_B\rho-\rho_{,C}[\gamma_{,A}\delta^C_B+\gamma_{,B}\gamma^C_A-\gamma_{,D}\bar{h}^{CD}\bar{h}_{AB}]$$
$$=-\rho F_C^{-2}[F_{C,A}F_{C,B}+\Omega_{C,A}\Omega_{C,B}-\bar{h}_{AB}\bar{h}^{CD}\times(F_{C,C}F_{C,D}+\Omega_{C,C}\Omega_{C,D})/2]/2. \quad (16)$$

Find out the new solution by the dual solution K_H of K_C is

$$d\hat{s}^2=\rho(dt+x^2d\phi)^2/F_H-F_H[e^{2\hat{\gamma}}\bar{h}^{AB}dx^Adx^B+\rho^2d\phi^2]/\rho, \quad (17)$$

$\hat{\gamma}$ is determined by the following equation

$$\begin{aligned}&\bar{\nabla}_A\bar{\nabla}_B\rho-\rho_{,C}[\hat{\gamma}_{,A}\delta^C_B+\hat{\gamma}_{,B}\delta^C_A-\hat{\gamma}_{,D}\hat{\bar{h}}^{CD}\hat{\bar{h}}_{AB}]\\&=-\frac{\rho^{-1}F_H^2}{2}\left[\left(\frac{\rho}{F_H}\right)_{,A}\left(\frac{\rho}{F_H}\right)_{,B}+\frac{\rho^2}{F_H^4}\Omega_{H,A}\Omega_{H,B}\right.\\&\left.\quad-\hat{\bar{h}}_{AB}\hat{\bar{h}}^{CD}\left(\left(\frac{\rho}{F_H}\right)_{,C}\left(\frac{\rho}{F_H}\right)_{,D}+\frac{\rho^2}{F_H^4}\Omega_{H,C}\Omega_{H,D}\right)\Big/2\right].\end{aligned} \quad (18)$$

The field equation remained at last is (12), and they can be eventually expressed as

$$\begin{aligned}&K(J)_{,11}K(J)_{,22}-(K(J)_{,12})^2=\Delta^2(J),\\&R(J)K(J)_{,11}+2SK(J)_{,22}+TK(J)_{,22}=0\end{aligned} \quad (19)$$

where

$$\begin{aligned}&R(J)=2[F^2(J)(\ln\Delta(J)_{,2}]_{,2}+4J^2,\\&S(J)=[F^2(J)(\ln\Delta(J))_{,1}]_{,2}+[F^2(J)(\ln\Delta(J)_{,2}]_{,1},\\&T(J)=2[F^2(J)(\ln\Delta(J))_{,1}]_{,1}.\end{aligned} \quad (20)$$

If Δ is a known function, superpotential $K(J)$ can be got from (19). Formula (19) is Monge–Ampere [5, 6] equation. The ordinary Monge–Ampere equation is

$$U\{K_{,11}K_{,22}-(K_{,12})^2]+RK_{,11}+2SK_{,12}+JK_{,22}+V=0 \quad (21)$$

when $J=i$, the above results can all be converted into the results of Cox; when $J=\epsilon$, new result can be got. It corresponds to the new superpotential K_H and the new gravitational field solution $(\hat{f},\hat{\omega},\hat{\gamma})$. Discussing how to get K_H from the known K_C, i.e. how to generate new solutions of gravitational fields, is another context of this paper.

3 The Generation and Application of the Dual Superpotential

Lemma 1 *If $G(x,y)$ and $G(x,iy)$ are both real numbers, and analytic at x, y, then*

$$\begin{aligned}&\partial G(x,iy)/\partial x=U_y(\partial G(x,y)/\partial x),\\&\partial G(x,iy)/\partial y=iU_y(\partial G(x,y)/\partial y).\end{aligned} \quad (22)$$

Where transformation U_y is: $H(x,y)\to\tilde{H}(x,y)=H(x,iy)$. The above equation can be written as

$$\begin{aligned}&\partial_x(U_y(G))=U_y(\partial_xG),\\&\partial_y(U_y(G))=iU_y(\partial_yG).\end{aligned} \quad (23)$$

Deduction 1 $\partial_y G(x, iy)$ *must be purely imaginary.*

We can get the following results by using of the above transformation: If $K_C(x^1, x^2)$ is a real solution of field equation (19), and also it satisfies the condition: $K_C(x^1, ix^2)$ is also real, then $K_H(x^1, x^2) = K_C(x^1, ix^2)$ is a (H) real solution of (19), that is, dual solution of K_C. We need to prove two sides: first, K_H is really a (H) solution; second, $[K_C, K_H]$ is a dual real number couple.

Proof use the above transformation U_{x^2}, we can deduce that

$$\begin{aligned}
&K_{H,1} = U_{x^2}(K_{C,1}), \qquad K_{H,2} = iU_{x^2}(K_{C,2}),\\
&K_{H,11} = U_{x^2}(K_{C,11}), \qquad K_{H,12} = iU_{x^2}(K_{C,12}) = K_{H,21},\\
&K_{H,22} = -U_{x^2}(K_{C,22}).
\end{aligned} \tag{24}$$

$$\begin{aligned}
R_H &= 2\{U_{x^2}(F_C^2)[U_{x^2}(\ln \Delta_C)]_{,2}\}_{,2} - 4 = 2i\{U_{x^2}(F_C^2)U_{x^2}(\ln \Delta_C)_{,2}\}_{,2} - 4\\
&= 2i\{U_{x^2}(F_C^2)(\ln \Delta_C)_{,2}\} - 4 = -2\{U_{x^2}[(F_C^2)(\ln \Delta_C)_{,2}]_{,2} - 4\} = -U_{x^2}(R_C).
\end{aligned} \tag{25}$$

Similarly, we have

$$\begin{aligned}
S_H &= iU_{x^2}(S_C),\\
T_H &= U_{x^2}(T_C).
\end{aligned} \tag{26}$$

Hence,

$$\begin{aligned}
&R_{11}K_{H,11} + 2S_H K_{H,12} + T_H K_{H,22}\\
&\quad = -U_{x^2}(R_C)U_{x^2}(K_{C,11}) + 2[iU_{x^2}(S_C)][iU_{x^2}(K_{C,12})] + U_{x^2}(T_C)U_{x^2}(K_{C,22})\\
&\quad = -U_{x^2}(R_C K_{C,11} + 2S_C K_{C,12} + T_C K_{C,22}) = -U_{x^2}(0) = 0.
\end{aligned} \tag{27}$$

We can see that K_H is really a real number, which in fact is very naturally, because the double Ernst equation guarantees so. The significance of this theorem lies in that under some conditions ($K_C(x^1, ix^2)$ can still be real), solve the Ernst dual solution, which means N-K substitution and analytic continuation can be accomplished by purely imaginary substitution U_{x^2}, after solving real K_C equation and finding superpotential. We still should prove that $K(J)$ is surely a double real function (i.e.

$$K(x^1, x^2; J) = K_{(J)}(x^1, Jx^2), \qquad K_C = K_{(J)}(J = i), \qquad K_H = K_{(J)}(J = \epsilon)). \quad \square$$

Lemma 2 *If analytic functions $G(x)$ and $G(ix)$ are both real, $G(\epsilon x)$ is also real. Actually, the exponents of x in the expanded formula are all even numbers. Hence, $C(Jx)$ is a double real function.*

Deduction 2 *If the form of K_C is like $K_C(x^1, x^2) = G(x^1, (x^2)^2)$, $K_H = G(x^1, -(x^2)^2)$ is the dual solution.*

Below we will give the specific applications of the above methods:
(1) *Hoffman* solution, choose ρ as a constant

$$\rho = \rho_0 = \text{const}, \qquad \Delta = \rho_0 f^{-2} = \rho_0/[x^1 - (x^2)^2], \qquad K_C = 2\rho_0/[x^1 - (x^2)^2], \tag{28}$$

its dual solution is

$$\rho = \rho_0 = \text{const}, \qquad \Delta = \rho_0/[x^1 + (x^2)^2], \qquad K_H = 2\rho_0/[x^1 + (x^2)^2]. \tag{29}$$

(2) *Keer* solution.

$$\begin{aligned} K_C &= -a[\Omega_C^2 + (1 - f^2)] - (4m^2\Omega_C^2/a[\Omega_C^2 + (1-f)^2]^{-1} \\ &= -a[(x^2)^2 + (1 - \sqrt{x^1 - (x^2)^2})^2] - 4(m^2(x^2)^2/a)[(x^2)^2 + (1 - \sqrt{x^1 - (x^2)^2})^2]^{-1}, \end{aligned} \tag{30}$$

$$K_H = -a[-(x^2)^2 + (1 - \sqrt{x^1 + (x^2)^2})^2] + 4(m^2(x^2)^2/a)[-(x^2)^2 + (1 - \sqrt{x^1 + (x^2)^2})^2]^{-1}. \tag{31}$$

These results are the same as those we get from (19) directly.

(3) When $\Delta = \text{const}$, no real roots exist in the case discussed by Cox–Kinnersley; yet there exists dual real solutions in our case. (x^1, x^2) is used to simplify (x, y). Since $\Delta = \text{const}$, then $R(J) = J^2 4$, $S(J) = T(J) = 0$, the field equation (26) changes into

$$\begin{aligned} &K(J)_{,11} = 0, \\ &K(J) = a(y; J)x + b(y; J). \end{aligned} \tag{32}$$

We find that $\Delta_C^2 = -[\partial a_C(y, J)/\partial y]$ have no real solutions. While

$$\begin{aligned} \Delta_H^2 &= [U_y(K_C)]_{,11}[-U_y(K_C)]_{,22} - [iU_y(K_C)]_{,12}^2 \\ &= -U_y(K_C)_{,11}(K_C)_{,22} + U_y(K_{C,12})^2 = -U_y(\Delta_C^2) \\ &= +U_y[\partial a_C(y, J)/\partial y]^2. \end{aligned} \tag{33}$$

Generally , We choose a as a single real function, that is

$$a(y, J) \equiv a(y) = \Delta y + b(y). \tag{34}$$

Let $\Delta = 1$, $K_H = xy + b(y)$, so we get real solutions

$$\begin{aligned} &\rho_H = \Delta \hat{f} = x^2 + y^2, \\ &\bar{h}_{AB} = \begin{bmatrix} 1 & 1 \\ 0 & B \end{bmatrix}, \\ &B = b''(y). \end{aligned} \tag{35}$$

Define $F(J) = \sqrt{x + J^2 y^2}$, $\Omega(J) = y$, then

$$\begin{aligned} &\omega = K_{C,1}, \qquad f = \sqrt{x - y^2}, \\ &\hat{\omega} = y, \qquad \hat{f} = \rho_H F_H^{-1} = \Delta_H F_H^{-1} = \rho_H/\sqrt{x + y^2}. \end{aligned} \tag{36}$$

Hence, we can get that $K(J)$ corresponds to

$$\begin{aligned} &\Delta(J) = \sqrt{\det(K_{,AB})}, \qquad \rho(J) = \Delta(J) \cdot F^2(J), \qquad \bar{h}_{AB}(J) = \Delta^{-1}(J)K(J)_{,AB}, \\ &\bar{h}^{AB}(J) = \Delta^{-1}(J)\epsilon^{AC}\epsilon^{BD}K(J)_{,CD}. \end{aligned}$$

At last, we get dual line element as follows:

$$\begin{aligned} ds^2 &= f(dt - K_{C,1}d\phi)^2 - f^{-1}[e^{2\gamma}\bar{h}_{AB}dx^A dx^B + \rho^2 d\phi^2],\\ d\hat{s}^2 &= \hat{f}(dt - x^2 d\phi)^2 - \hat{f}^{-1}[e^{2\hat{\gamma}}\hat{\bar{h}}_{AB}dx^A dx^B + \hat{\rho}^2 d\phi^2]. \end{aligned} \tag{37}$$

Then, the line element got in the situation of $\Delta = \text{const}$ is

$$\begin{aligned} d\hat{s}^2 &= \hat{f}(dt + y d\phi)^2 - \hat{f}^{-1}[e^{2\hat{\gamma}}(dxdy + Bdy^2) + \hat{f}^4 d\phi^2],\\ F_H^2 &= x + y^2, \qquad \hat{f} = \rho/F_H,\\ \hat{\gamma} &= \ln(F_H^2)/16 + \ln(B + 4y)/2 - \int [B(y) + 4y]^{-1}(dy/2). \end{aligned} \tag{38}$$

4 Conclusion

Now we can see that why there exists no real solutions in the paper written by Cox. As from the view of N-K theory, the dual solution of $d\hat{s}^2$ should be got by analytic continuation of $(\hat{\rho}F_H^{-1}, i\psi) = (\hat{\rho}/\sqrt{x^1, (x^2)^2}, ix^2)$. However, the analytic continuation of $(\hat{\rho}/\sqrt{x^1, (x^2)^2}, ix^2)$ can not guarantee the real quality of Δ, which meantime is a specific case showing that some ordinary Ernst equations have no concrete real solutions while have dual solutions instead.

References

1. Kunstatter, G., Moffat, J.W., Malzan, J.: J. Math. Phys. **24**, 886 (1983)
2. Cox, D., Kinnersly, W.: J. Math. Phys. **20**, 1225 (1979)
3. Zhong, Z.-Z.: J. Math. Phys. **26**, 2589 (1985)
4. Zhong, Z.-Z.: J. Math. Phys. **26**, 404 (1985)
5. Neugebauer, G., Kramer, D.: Ann. Phys. **24**, 62 (1969)
6. Courant, R., Hibert, D.: Methods of Mathematical Physics, vol. II. Wiley, New York (1966)